Pascal's Triangle and the Binomial Theorem

Blaise Pascal (1623 - 1662) was a French mathematician, philosopher, and finally, religious mystic who made seminal contributions to the theory of probability, to algebra, and posthumously, to Newton's "new math": what we know today as the differential calculus. He is probably most famous for "Pascal's Triangle", illustrated below.

$$\begin{array}{ccccccccccccccccccc}
&&&&&&&&&1&&&&&&&&& \\
&&&&&&&&1&&1&&&&&&&& \\
&&&&&&&1&&2&&1&&&&&&& \\
&&&&&&1&&3&&3&&1&&&&&& \\
&&&&&1&&4&&6&&4&&1&&&&& \\
&&&&1&&5&&10&&10&&5&&1&&&& \\
&&&1&&6&&15&&20&&15&&6&&1&&& \\
&&1&&7&&21&&35&&35&&21&&7&&1&& \\
&1&&8&&28&&56&&70&&56&&28&&8&&1& \\
1&&9&&36&&84&&126&&126&&84&&36&&9&&1
\end{array}$$

Given the first two rows, we can construct the third row as 1, then the sum of the items to the left and to the right of our current position in the row above us, and finishing with another 1. Now repeat this pattern for every subsequent row. So in row 4, $1+2=3$, then $2+1=3$; in row 5, $1+3=4$, $3+3=6$, $3+1=4$; and so on.

If we add up all the numbers on each row we see this pattern emerging: $1, 2, 4, 8, \ldots$. In other words, successive powers of two: 2^0, 2^1, 2^2, 2^3, $\ldots$. After we learn about the Binomial Theorem we will understand why this is true.

So what caused Pascal to dream up this triangle? Well, there were a couple of problems he worked on 350 years ago that led him to it. He was interested in a gambling problem, and he wrote letters back and forth with Pierre de Fermat discussing probability theory.

Let's suppose we are flipping a perfectly fair coin, so there's exaactly a 50% chance that it comes up heads or tails each time we flip it. What is the chance that heads will come up exactly 3 times in 4 flips? Well there are 16 possible outcomes for four flips of the coin; 2 possibilities each time, and $2^4 = 16$. Let's list them.

HHHH	HHHT	HHTH	HHTT	HTHH	HTHT	HTTH	HTTT
THHH	THHT	THTH	THTT	TTHH	TTHT	TTTH	TTTT

Here HHHH means four heads in a row, HHHT means three heads, then tails once, etc. Four of these sequences – HHHT, HHTH, HTHH, and THHH – contain exactly 3 heads.

So there are four chances out of sixteen, or one out of four, that we'll get exactly 3 heads out of 4 flips of the perfectly fair coin.

Let's regroup the possible outcomes so that the case with four heads comes first, then the cases with two heads, then one head, and finally all tails. It looks like this.

HHHH HHHT HHTH HTHH THHH HHTT HTHT HTTH
THHT THTH TTHH HTTT THTT TTHT TTTH TTTT

So we can see there is one case with all heads, four cases with three heads, six cases with two heads, four cases with one head, and one case with no heads. 1, 4, 6, 4, 1 is the fifth row (row #4 if we start counting from 0) in the triangle on page one, above.

Instead of heads and tails, we might just as well have used 1s and 0s to write out the four-bit binary numbers from 0000 to 1111. Or we could use any two symbols whatsoever. If we have two different symbols and we're asked how many ways we can write a sequence of n symbols so that one particular symbol occurs exactly k times, the answer will always be found in the nth row and the kth column of Pascal's Triangle, so long as we call the very first row "row number zero", and the first column in each row "column number zero".

Let's now define a symbol $\binom{n}{k}$ called *n choose k* that represents the value in the nth row and kth column of Pascal's Triangle, where we start counting rows from row 0 and we count columns starting with column 0. Let us also define the *factorial function* (denoted by $n!$) like this: $0! = 1! = 1$; for all subsequent natural numbers n, $n! = n * (n-1)!$. So $2! = 2 * 1! = 2 * 1 = 2$, $3! = 3 * 2! = 3 * 2 = 6$, $4! = 4 * 3! = 4 * 6 = 24$, and so forth. I assert that

$$\binom{n}{k} = \frac{n!}{k!(n-k)!} = \frac{n * (n-1) * (n-2) * \ldots * (n-k+1)}{k!} \quad (k \leq n)$$

Here is a proof of the assertion. Think about flipping a coin n times and counting heads. Let's start with $\binom{n}{0}$ (n flips, no heads). Clearly there is only one way this can happen; the coin has to come up tails every time. Applying the formula, $\binom{n}{0} = \frac{n!}{0! * n!} = \frac{n!}{n!} = 1$. So that's OK.

For the case $k = 1$, it is clear that heads will come up once if and only if tails come up $(n-1)$ times. But this can happen in n different ways: a head on the first flip, a head on the second flip, ..., a head on the nth flip. And $\binom{n}{1} = \frac{n!}{1! * (n-1)!} = n$.

How many ways can we get two heads? Well, for each of the n sequences with one head and $(n-1)$ tails, we can change one of the tails into a head. But if, for instance, we add the second head to the sequence HTTTTT... in the second position, obtaining HHTTTTT..., when we add a second head to the second sequence with one head, THTTTTT..., we may obtain HHTTTTT... again. So we have to divide by two, to eliminate double counting. So we can place the first head in any one of n positions, and the second head in any one of $n-1$ positions, divided by two to eliminate the double counting, giving $n * (n-1)/2$ unique sequences with two heads. And sure enough, $\binom{n}{2} = \frac{n!}{2! * (n-2)!} = \frac{n * (n-1)}{2}$.

Similar reasoning applies when there are three, four, or even more heads. If we have already inserted k heads into a sequence of length n, there are $n-k$ ways to insert one more. But this process will produce $(k+1)!$ identical sequences, so we need to multiply the preceding value $\binom{n}{k}$ times $(n-k)$, then divide by $(k+1)$, to arrive at $\binom{n}{k+1}$. But that is exactly what the general formula for “n choose k” tells us to do. ◎

Binomial Expressions in Algebra

In algebra, any expression that contains two terms is called a binomial. An expression that contains at least two terms is, in general, a polynomial. So a binomial is the simplest, or shortest, polynomial. Typical binomials include $(x+y)$, $(4*x+2)$, $(1+1)$, and $(3*x^3-7*x^2)$.

A problem arising frequently in algebra is how to calculate some quantity like $(x+y)^n$. When n is small it's not too bad: $(x+y)^2 = (x+y)*(x+y) = x*x+x*y+y*x+y*y = x^2+2*x*y+y^2$. And $(x+y)^3 = (x+y)*(x+y)^2 = (x+y)*(x^2+2*x*y+y^2) = x*x^2+x*2*x*y+x*y^2+y*x^2+y*2*x*y+y*y^2 = x^3+3*x^2*y+3*x*y^2+y^3$. But just try doing the algebra the hard way if you're asked to express $(x+y)^{100}$ as a polynomial $x^{100}+\ldots$!

Now the problem of expanding an algebraic expression like $(x+y)^n$ is entirely analogous to counting heads and tails in a sequence of n flips of a coin. Clearly we can write

$$(x+y)^n = \binom{n}{0}x^n + \binom{n}{1}x^{n-1}y + \binom{n}{2}x^{n-2}y^2 + \ldots + \binom{n}{n}y^n \tag{1}$$

At this point it will be helpful to introduce a sort of shorthand for extended sums. We use the Greek letter Σ to indicate summation, like this:

$$\sum_{k=0}^{n}(a_k) = a_0 + a_1 + a_2 + \ldots + a_n \tag{2}$$

Now we can express the Binomial Theorem compactly, and rewrite equation (1) above this way:

$$(x+y)^n = \sum_{k=0}^{n}\binom{n}{k}x^{n-k}y^k \tag{3}$$

Now we can easily see why the sum of all the numbers on each row in Pascal's Triangle is equal to a power of two: setting $x=1$ and $y=1$ in formula #3 we obtain the sum

$$(1+1)^n = 2^n = \binom{n}{0} + \binom{n}{1} + \binom{n}{2} + \ldots + \binom{n}{n-1} + \binom{n}{n}$$

because all the expressions $x^{n-k}y^k$ in formula #3 become $1^{n-k}1^k = 1$, leaving us with just the sum of all the binomial coefficients on row n of Pascal's Triangle.

Extending the Binomial Theorem to Fractional Powers

In 1665, when he was only 22 years old, Isaac Newton was able to show that, with suitable restrictions, the Binomial Theorem is true for fractional exponents. When used this way, the formula produces an infinite sequence of terms which may converge to a limit. One must also take the factor $(n-k)!$ out of the denominator and express the binomial coefficient this way:

$$\binom{n}{k} = \frac{n(n-1)(n-2)\ldots(n-k+1)}{k!}$$

When $n = 1/2$, the first few binomial coefficients are 1, $1/2$, $(1/2)(-1/2)/2$, $(1/2)(-1/2)(-3/2)/3!$, $(1/2)(-1/2)(-3/2)(-5/2)/4!$, $(1/2)(-1/2)(-3/2)(-5/2)(-7/2)/5!, \ldots$ which simplifies to $1, (1/2)$, $(-1/8)$, $(1/16)$, $(-5/128)$, $(7/256), \ldots$:

$$(1+x)^{(1/2)} = \sqrt{(1+x)} = 1 + \frac{1}{2}x - \frac{1}{8}x^2 + \frac{1}{16}x^3 - \frac{5}{128}x^4 + \frac{7}{256}x^5 + \ldots \tag{4}$$

If we set $x = 1$ in formula #4, above, we find that the series converges, but very slowly: the first six terms give the approximation $\sqrt{2} = 1.4257\ldots$, which is only good to two decimal places. But if we cleverly set $x = -1/2$, we get an estimate for $\frac{\sqrt{2}}{2}$ that is correct to 3 places. And if we set $x = -1/9$ the estimate for $\frac{\sqrt{8}}{3}$ is correct to seven decimal places.

What's the point? Well, 350 years ago, in Newton's day, nobody had a digital computer or a hand-held electronic calculator. All numerical calculations had to be done by hand, with pen and ink. A number like $\sqrt{2}$ was important; it comes up a lot in trigonometry, which is still used today for surveying, and astronomy, and GPS satellites. So if someone could cut back on the labor involved in computing the values of mathematical functions, everybody benefitted. Power series, that is, expressions of the form $a_0 + a_1x + a_2x^2 + a_3x^3 + \ldots$, are very important in higher mathematics. And Newton's generalization of the Binomial Theorem to include fractional exponents was an important step in man's understanding of power series.

Euler's Number e and the Binomial Theorem

The base of natural logarithms, $e = 2.7182818284590 45\ldots$ is, after π, the most famous irrational number in all of mathematics. Although Jacob Bernoulli invented it, it was Leonhard Euler, a Swiss mathematician who came along about 50 years later, who made it famous. We call it Euler's number to honor Euler, who discovered the most famous equation in all of mathematics:

$$e^{i\pi} + 1 = 0$$

Let's explore, briefly, how this beautiful formula relates to the Binomial Theorem, and sketch out the process that led Euler to discover it. The first step is to define the number e. The classical approach is to say that e is the limit, as n approaches ∞, of $(1 + \frac{1}{n})^n$. Or, in symbols:

$$e = \lim_{n\to\infty} \left(1 + \frac{1}{n}\right)^n$$

What, exactly, does this mean? Well, it means that if we form the sequence 2^1, $(3/2)^2$, $(4/3)^3$, $(5/4)^4$, $\ldots$ the result will eventually get arbitrarily close to some limiting value which we call e. And indeed the sequence 2, $9/4 = 2.25000$, $64/27 = 2.37037\ldots$, $625/256 = 2.44140\ldots$ does seem to be closing in on a limiting value somewhere in between 2 and

3. Let's use the Binomial Theorem to see if we can get a more precise value a bit more quickly than by raising say $101/100$ to the one hundredth power.

When we expand the expression $(1+\frac{1}{n})^n$ using the Binomial Theorem, we can omit all the factors $1^{(n-k)}$ because 1 to any power is just 1. We have

$$\begin{aligned}\left(1+\frac{1}{n}\right)^n &= \sum_{k=0}^{n}\binom{n}{k}\left(\frac{1}{n}\right)^k \\ &= \binom{n}{0}+\binom{n}{1}\left(\frac{1}{n}\right)+\binom{n}{1}\left(\frac{1}{n}\right)^2+\dots \\ &= 1+1+\sum_{k=2}^{n}\binom{n}{k}\left(\frac{1}{n}\right)^k\end{aligned}$$

where we have simplified the sum by noticing that the first two terms are equal to 1 for every n. Now let's examine the third term in the series, $\binom{n}{2}(\frac{1}{n})^2 = \frac{n(n-1)}{2!n^2}$. What happens to this fraction as n grows very large? Well, it will approach the limit $\frac{1}{2!}$, because $\frac{n^2-n}{n^2} = 1-\frac{1}{n}$ will approach the limit 1 when n grows without bound.

The same reasoning applies to all the remaining terms in the series: when n is very large, $\binom{n}{k}\frac{1}{n^k}$ will approach $\frac{1}{k!}$. We conclude that

$$e = \lim_{n\to\infty}\left(1+\frac{1}{n}\right)^n = \sum_{k=0}^{\infty}\frac{1}{k!} \tag{5}$$

How does all this relate to Euler and $e^{i\pi}$? Well, formula #5 doesn't just define a number; it's a prototype for a power series that defines the *exponential function*. I'm not going to get into all the whys and wherefores right now, or try to define absolute convergence in the abstract. I think you can understand how Euler came up with his famous formula without learning all about the calculus. Here's the power series that defines the exponential function everywhere throughout the complex plane.

$$\exp(z) = e^z = \sum_{k=0}^{\infty}\frac{z^k}{k!} \tag{6}$$

It's time for a quick word about notational conventions. In mathematical analysis, x and y are generally used to represent variables that assume real values only. A special symbol i is used to represent the *imaginary unit*. We write $i=\sqrt{-1}$; $i^2=-1$, by definition. Clearly i cannot be a real number, because $x^2 \geq 0$ for every real number x. We can perform algebraic operations on complex numbers of the form $x+iy$ by simply following the usual rules for manipulating algebraic expressions and remembering that $i^2=-1$. Traditionally a variable z is used to denote a complex variable $x+iy$. That's why I wrote equation $\#6$ with a z, to emphasize the fact that $\exp(z)$ is a function of a complex variable.

Let's follow Mr. Euler on his journey of discovery. He began by asking "What will happen if z is a purely imaginary number?" Well, let's make the substitution $z=ix$, where i is the imaginary unit and x is a real number, and rewrite equation $\#6$.

$$\begin{aligned}
e^{ix} &= \sum_{k=0}^{\infty} \frac{(ix)^k}{k!} && (7)\\
&= 1 + \frac{(ix)}{1!} + \frac{(ix)^2}{2!} + \frac{(ix)^3}{3!} + \frac{(ix)^4}{4!} + \frac{(ix)^5}{5!} + \frac{(ix)^6}{6!} + \frac{(ix)^7}{7!} \dots && (8)\\
&= 1 + i * \frac{x}{1!} - \frac{x^2}{2!} - i * \frac{x^3}{3!} + \frac{x^4}{4!} + i * \frac{x^5}{5!} - \frac{x^6}{6!} - i * \frac{x^7}{7!} \dots && (9)\\
&= 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + - \dots + i * \left(\frac{x}{1!} - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + - \dots \right) && (10)
\end{aligned}$$

where in equation 8 we have written out the first eight terms of the series explicitly; in equation 9 we have substituted $i^2 = -1$, $i^3 = -i$, $i^4 = 1$, $i^5 = i$, etc.; and in equation 10 we have separated the even and odd powers of x into two series, and factored i out of the second series. (I should insert a note of caution here. It's not always OK to rearrange the terms in an infinite series, as I have done here. In this case it's kosher; trust me.)

And now Mr. Euler had an aha! moment. He recognized the separated series in $\#10$ from his earlier work with trigonometric functions:

$$\begin{aligned}
\cos x &= 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + - \dots = \sum_{k=0}^{\infty} (-1)^k \frac{x^{2k}}{(2k)!}\\
\sin x &= x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + - \dots = \sum_{k=0}^{\infty} (-1)^k \frac{x^{2k+1}}{(2k+1)!}
\end{aligned}$$

And so he derived his famous equation as a special case of a more general relationship:

$$\begin{aligned}
e^{ix} &= \cos x + i \sin x\\
e^{i\pi} &= -1
\end{aligned}$$

because $\cos \pi = -1$, and $\sin \pi = 0$. Pretty slick, eh?