# Limits and Real Numbers

The *limit* of a sequence (or the limit approached by a function) is an important concept in higher mathematics. My intention today is to define the notion of a limit, and to give a few examples that will help you understand what a limit is. But first we should be sure we understand the various kinds of numbers: natural numbers, integers, rational numbers, real numbers, and complex numbers. People have defined some additional kinds of numbers (for example, the surreal numbers, and quaternions), but we won't concern ourselves with those. The integers, the rationals, the reals, and the complex numbers are enough to support what I would call classical mathematical analysis – roughly, the mathematics of quantitative analysis that was discovered from ancient times through the mid-nineteenth century.

So what is a number? The simplest numbers are the natural numbers, or counting numbers: $0$, $1$, $2$, etc. We denote the set of all the (infinitely many) natural numbers as $\mathbb{N}$. The *integers* are closely related to the natural numbers: the integers are the natural numbers plus their additive inverses (negatives). We denote the set of all integers as $\mathbb{Z}$. Symbolically, $\mathbb{Z} = \{\ldots, -3, -2, -1, 0, 1, 2, 3, \ldots\}$. We can visualize the integers as being equally-spaced points lying on a number line stretching from $-\infty$ to $+\infty$.

We observe that $\mathbb{Z}$ is closed under the three arithmetic operations known as addition, subtraction, and multiplication, which is to say that for any two integers $m$ and $n$, $m + n$, $m - n$, and $m * n$ are integers. The situation is different when it comes to division. The quantity $m/n$ where $n \neq 0$ may be an integer (if $n$ divides $m$) or it may not be an integer (when dividing $m$ by $n$ leaves a remainder $r \neq 0$). This leads us to define a new kind of number, known as a rational number, and commonly called a fraction. We denote the set of all rational numbers with the symbol $\mathbb{Q}$. $\mathbb{Q} =$ the set of all proper and improper fractions $\frac{m}{n}$ where $m$ is any integer whatsoever, and $n$ is any integer, excluding $0$.

The field of rational numbers $\mathbb{Q}$ is closed under all four arithmetic operations, as can easily be shown (try writing out a proof for yourself). The rational numbers were known to the ancient Egyptians, and to the Babylonians, and to the Greeks. For many years people thought the rational numbers were the only numbers that could possibly exist. But eventually some clever Pythagorean geometer realized that the Theorem of Pythagoras implies that the ratio of the length of the diagonal in a square to the length of its side cannot be a rational number (*cf* the proof that $\sqrt{2}$ is irrational). This discovery was the beginning of man's quest fully to understand the *real numbers*, a quest that is still going on today.

We know that there are some real numbers that correspond to constructible geometric entities, but how can we characterize the entire set of real numbers, usually denoted by $\mathbb{R}$? There are several ways to do this. The first way is simply to associate the real numbers with the points on a line stretching from $-\infty$ to $+\infty$. Every point on that line then corresponds to a particular real number, and every real number corresponds to a single point on that line. This is intuitively appealing, but gives us little additional insight into the nature of the real number system.

Please take note of a peculiar property of $\mathbb{Q}$, the set of all rational numbers. We can find a rational number $s$ lying in between any two rational numbers $t$ and $u$, no matter how close $t$ is to $u$: just form the average $\frac{t+u}{2}$ and voila! So there is no "nearest neighbor" to a particular rational number. We say that the set of numbers $\mathbb{Q}$ is topologically dense. While the set $\mathbb{Q}$ is dense, it is not topologically complete. The real numbers $\mathbb{R}$ are said to be the topological completion of $\mathbb{Q}$ (whatever that means; it should become clear in a bit). To define the real numbers rigorously, we must introduce the notion of a *limit*.

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There are two distinct kinds of limits to think about. The first is the limit of a sequence $S$. (I'll address the other kind of a limit another day.) The notion of a limit only makes sense if $S$ contains infinitely many elements. The sequence $S$ is simply a set of numbers (for now, think of a sequence of rational numbers; we haven't yet formally defined a real number). In symbols, $S = \{s_1, s_2, s_3, \ldots\}$. A famous infinite sequence, known as the harmonic sequence because it comes up in music theory, is just the set $H = \{1, \frac{1}{2}, \frac{1}{3}, \frac{1}{4}, \ldots\}$.

An arbitrary infinite sequence is said to approach the limit $\ell$ if, for any $\epsilon > 0$ there exists an integer $L$ such that $|s_k - \ell| < \epsilon$ for every integer $k \geq L$. Symbolically we write $\lim_{k\to\infty} s_k = \ell$. Intuitively this means that the difference between $s_k$ and $\ell$ becomes arbitrarily small as $k$ approaches infinity. As an example, we can prove that the harmonic sequence converges to the limit $0$ by simply choosing an integer $L > \frac{1}{\epsilon}$ and applying the definition of a limit.

I'm going to digress a little here and insert an historical note. Sir Isaac Newton devised the differential and integral calculus sometime during the mid-seventeenth century. In Germany, Gottfried Wilhelm Leibniz made a similar discovery at about the same time. Calculus took the world by storm. Many formerly intractable engineering problems were easily solved using Newton's "new math". Mathematicians throughout Europe eagerly applied the new methods to all kinds of problems for almost 200 years without anyone bothering to ask "What, precisely, are the numbers produced by calculus, in a set-theoretic sense?" Guys like the Bernoulli brothers, and Leonhard Euler, and Karl Friedrich Gauss hurled infinitesimal quantities around with great abandon, but never bothered to define precisely what kind of numbers the calculus deals with.

The first mathematician to try to put Newton's calculus on a logically rigorous basis was Augustin-Louis Cauchy, a French mathematician born in 1789, on the cusp of the French Revolution. Cauchy defined what we now know as a Cauchy sequence this way: a sequence $S = \{s_i\}$ is Cauchy if, and only if, all the differences $|s_j - s_{j+k}|$ are less than any given $\epsilon > 0$, provided only that $j$ is chosen large enough. In other words, the terms of a Cauchy sequence are clustered around some limiting value. We can now define the real numbers $\mathbb{R}$ topologically, as the set of all the limits of Cauchy sequences in $\mathbb{Q}$.

There is another way to define the real numbers in terms of the rational numbers. This definition was devised by Richard Dedekind, a nineteenth-century German mathematician. Dedekind reasoned this way. Let us divide the set of rational numbers $\mathbb{Q}$ into a left set $A_L$ and a right set $A_R$ according to this rule: every number in $A_L$ is strictly less than every element of $A_R$. Because $\mathbb{Q}$ is dense, $A_L$ cannot have a greatest element. If $A_R$ has a smallest element $q$, we associate this particular division of $\mathbb{Q}$ with the rational number $q$; if $A_R$ has no smallest element, we associate this particular division of $\mathbb{Q}$ with an irrational real number $r$ that lies in between the elements of $A_L$ and $A_R$. We say that every such division of $\mathbb{Q}$ into a left set and a right set defines a *Dedekind cut*, and we define the set of real numbers $\mathbb{R}$ to be the set of all possible Dedekind cuts.

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At this point we should probably think about a few different kinds of Cauchy sequences. A sequence is said to be monotone if it is either non-increasing, or non-decreasing. In symbols, the sequence $S = \{s_i\}$ is monotone non-increasing if and only if $s_{i+1} \leq s_i$ for every $i$; similarly, $S$ is monotone non-decreasing if and only if $s_{i+1} \geq s_i$ for every $i$. The harmonic sequence $H = \{1, \frac{1}{2}, \frac{1}{3}, \ldots\}$ is monotone decreasing because each item in the sequence is strictly less than its predecessor. Subtracting each item in $H$ from $1$ yields the monotone increasing sequence $\{0, \frac{1}{2}, \frac{2}{3}, \frac{3}{4}, \ldots\}$.

Here's another Cauchy sequence, from the Fibonacci numbers $\{1, 1, 2, 3, 5, 8, 13, \ldots\}$. $S = \{\frac{1}{1}, \frac{1}{2}, \frac{2}{3}, \frac{3}{5}, \frac{5}{8}, \frac{8}{13}, \ldots\}$. This sequence is not monotone; $1 > \frac{1}{2} < \frac{2}{3} > \frac{3}{5} < \ldots$; the terms are closing in on the limiting value alternately from above and below. This Cauchy sequence converges to the golden ratio $\phi$. The Fibonacci numbers themselves fall out of a regular continued fraction containing only the digit $1$:

$$\phi = \cfrac{1}{1 + \cfrac{1}{1 + \cfrac{1}{1 + \cfrac{1}{1 + \cfrac{1}{1 + \cfrac{1}{1 + \ddots}}}}}} \tag{1}$$

Continued fractions are used a lot in computer algorithms that calculate things like logarithms, or exponentials, or sin, cos, tan, etc. In general, continued fraction representations of such functions generate sequences that close in on the limit from both above and below. This makes them particularly useful for computerized algorithms, because one can easily determine how precise a particular approximate value is, and halt the calculation when the desired number of bits of accuracy have been ground out.

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This is getting a little long, so I'll just briefly describe and define the field of *complex numbers*. We write $\mathbb{C}$ to represent the set of complex numbers. We define the complex field $\mathbb{C}$ to be the algebraic closure of the real numbers $\mathbb{R}$ and one additional element $i$, known as the imaginary unit, or $\sqrt{-1}$. (In electronics, people sometimes write $j$ to represent $i$.) We perform addition and subtraction easily, by simply separating the real and imaginary parts: $(a+bi)+(c+di) = (a+c)+(b+d)i$. Multiplication is also straightforward: $(a+bi)*(c+di) = ac + (ad+bc)i + bdi^2 = (ac-bd) + (ad+bc)i$.

Division is a bit more complicated. Let's start calculating $(a+bi)/(c+di)$ by first rewriting $1/(c+di)$. Clearly, $(c-di)/(c-di) = 1$. But $(c-di)*(c+di) = c^2+d^2$, where $c^2+d^2$ is a real number. So we have

$$\frac{a+bi}{c+di} = (a+bi)\frac{1}{c+di} = (a+bi)\frac{c-di}{(c-di)(c+di)} = \frac{(a+bi)(c-di)}{c^2+d^2} = \frac{(ac+bd)+(bc-ad)i}{c^2+d^2}$$

So all four arithmetic operations are easily performed in $\mathbb{C}$ by simply applying the ordinary rules of algebra and remembering that $i^2 = -1$. The complex numbers are easily visualized as points in a plane, with the horizontal axis representing real numbers (imaginary part is zero) and the vertical axis representing imaginary numbers (real part is zero). Amazingly enough, mathematicians call this mental construct "the complex plane". Here's a picture.