Functions, Continuity, and Differentiability

One of the foundational concepts in mathematical analysis is the notion of a function. One might even say that the function of mathematical analysis – the differential calculus and related fields – is to facilitate and elucidate our understanding of functions.

So what is a function? A function is a rule that associates an input (called the *argument* of the function, usually a number), with an output (called the function's *value*). The function's value may be a single number, or several numbers, or even something more complicated, such as a vector. For now, we will restrict our attention primarily to single-valued functions of a real variable x. We write $f(x) = y$ to signify that y is a function of x. We call the set of input values x for which $f(x)$ is defined the *domain* of the function $f(x)$, and we call the set of possible output values y the *range* of the function $f(x)$.

Let's look at a concrete example, the simple parabola $f(x) = x^2 - 4$. Here's a picture.

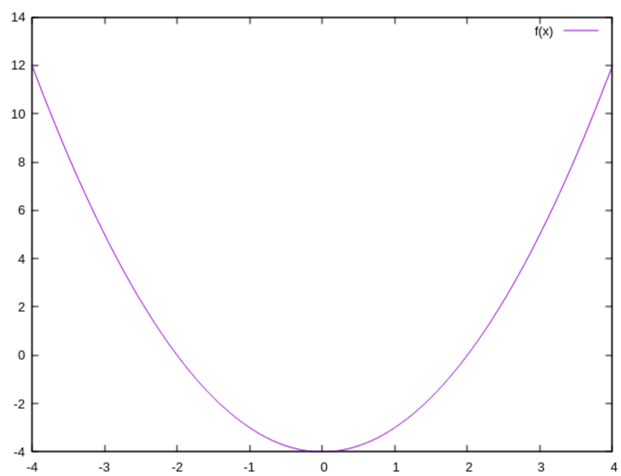

Figure 1: The function $f(x) = x^2 - 4$

Although we can't illustrate all of it here, it should be obvious that the domain of this function is the entire real number line $(-\infty, +\infty)$, and the range of this function is $[-4, +\infty)$. (Note the use of (and [– the domain of $f(x)$ is the open interval $(-\infty, +\infty)$; the range of $f(x)$ is the half-open interval $[-4, +\infty)$. We use ( ) to indicate that either end of an interval is open, and [ ] to indicate that one end, or both, are closed.)

A function $f(x)$ is said to be *continuous* at the point x_0 if and only if the value of $f(x)$ approaches $f(x_0)$ as x approaches x_0. Formally we define continuity in terms of limits: $f(x)$ is continuous at the point x_0 if and only if $\lim_{x \to x_0} f(x) = f(x)$. If we want to get really technical we can talk about a left-hand limit, or a limit from below (left-continuity), and a right-hand limit, or a limit from above (right-continuity). The left-hand limit is evaluated by considering an increasing sequence of numbers x_i such that $x_i < x_0$ for every x_i and $x_i \to x_0$ as $i \to \infty$. Conversely, the right-hand limit is evaluated using a decreasing sequence of numbers with similar characteristics ($x_i > x_0$ and $x_i \to x_0$ as $i \to \infty$). Usually, though, we'll just talk about continuity at x_0, by which we mean $x \to x_0$ from both sides.

At this point it behooves us to give a precise definition of what the statement

$$\lim_{x \to x_0} f(x) = f(x_0)$$

means. We say that $\lim_{x \to x_0} f(x) = f(x_0)$ if, and only if, for any given number $\epsilon > 0$, there exists a number δ_0 such that $|f(x_0 \pm \delta) - f(x_0)| < \epsilon$ provided only that $\delta < \delta_0$. (The vertical bars $||$ mean absolute value.) In other words, given a tiny positive number ϵ, it is always possible to find another tiny positive number δ_0 such that, for values of x within a distance δ_0 of x_0, the absolute value of $f(x) - f(x_0) < \epsilon$.

Let's look at a concrete example, to see how the definition of a limit is applied. Consider the function $f(x) = x^2$. I assert that this function is continuous at every point x_0 from $-\infty$ to $+\infty$. How so? Well, for any x_0, $|f(x_0 \pm \delta) - f(x_0)| = |(x_0 + \delta)^2 - x_0^2| = |x_0^2 \pm 2x_0\delta + \delta^2 - x_0^2| \leq |2x_0\delta + \delta^2|$. So by choosing δ so that $(\delta + \delta^2/2x_0) < |\epsilon/2x_0|$ we can satisfy the definition of approaching the limit $f(x_0)$, even for very large values of $|x_0|$.

Not all simple functions are continuous. For instance, consider the behavior of $f(x) = 1/x$ at the point $x = 0$. As we approach $x = 0$ from the left, $f(x)$ approaches $-\infty$; as x approaches 0 from the right, $f(x)$ approaches $+\infty$. See Figure 2, below.

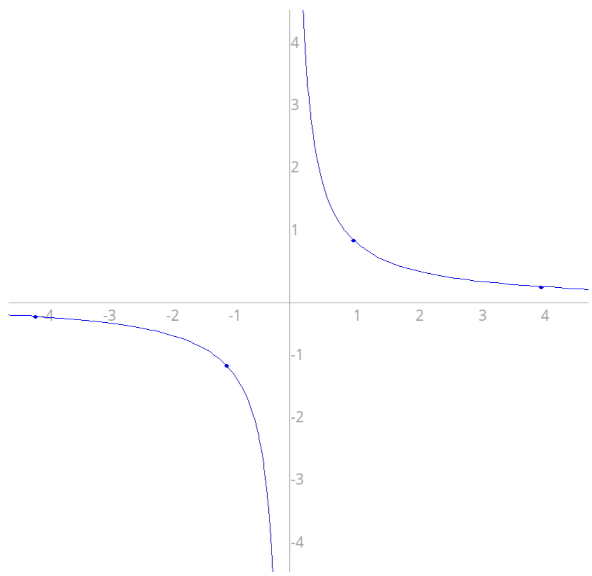

Figure 2: The function $f(x) = 1/x$

Introduction to Calculus: Finding the Derivative of a Polynomial

Calculus got its name because it allows us to perform some very complex calculations using mathematical functions with a modicum of effort. As its name suggests, calculus makes many mathematical calculations easy, by enunciating some simple, almost mechanical, rules. The first of these rules tells us how to calculate the instantaneous rate of change of any polynomial function $P(x) = a_0 + a_1x + a_2x^2 + a_3x^3 + \ldots + a_nx^n$ by generating an auxiliary polynomial $P'(x)$, which we call the *derivative* of P. Let's dive in and see how this is done.

Given a polynomial function $P(x)$, we seek a method to calculate the instantaneous rate of change at any point on the curve. To do that, we must first define exactly what we mean by "instantaneous rate of change." Let's start by defining the average rate of change of a function $f(x)$ over a small interval δ. The value of the function $f(x)$ at a particular point x is just $f(x)$; clearly, at a point $x + \delta$ the function assumes the value $f(x+\delta)$. So we can easily see that the average rate of change between $x+\delta$ and x is given by

$$\frac{f(x+\delta) - f(x)}{(x+\delta) - x} = \frac{f(x+\delta) - f(x)}{\delta}$$

Let's take a couple of concrete examples, with $x = 2$ and $\delta = 0.1$, just to be sure we have this down. If $f(x)$ is just a constant value a, the average rate of change is zero: $\frac{f(2.1)-f(2)}{2.1-2} = \frac{a-a}{0.1} = \frac{0}{0.1} = 0$. If $f(x) = x$, the average rate of change is 1: $\frac{f(2.1)-f(2)}{2.1-2} = \frac{2.1-2}{0.1} = \frac{0.1}{0.1} = 1$. If $f(x) = x^2$, the average rate of change is 4.1: $\frac{f(2.1)-f(2)}{2.1-2} = \frac{2.1^2-2^2}{0.1} = \frac{4.41-4}{0.1} = \frac{0.41}{0.1} = 4.1$. Here. You can work out the details for yourself. If $f(x) = x^3$, the average rate of change is 12.61. For $f(x) = x^4$, it's 48.62025. And so on.

With all that under our belts, we're ready to define the instantaneous rate of change of a function. We define the derivative, or instantaneous rate of change of $f(x)$,to be

$$f'(x) = \lim_{\delta \to 0} \frac{f(x+\delta) - f(x)}{\delta} \tag{1}$$

if, and only if, that limit exists. If the limit (1) exists at a particular point x_0, the function f is said to be *differentiable* at the point x_0. If the limit does not exist, the function f is non-differentiable. It is easily shown that every function that is differentiable at a point x_0 is also continuous at x_0, because the limit defined in equation (1) above cannot exist unless the numerator approaches 0 as δ approaches 0. But a continuous function is not necessarily differentiable at every point. Think about $f(x) = |x|$. That function changes direction abruptly at the point 0. It is continuous at 0. But it is not differentiable there. Approaching 0 from below, the rate of change of $|x|$ is -1; approaching 0 from above, the rate of change is $+1$. So $|x|$ does not have a well-defined derivative at 0.

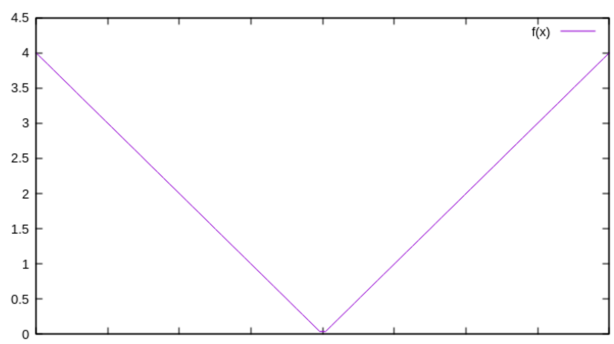

Figure 3: $f(x) = |x|$ is not differentiable when $x = 0$

So now let's return to the problem of differentiating the general polynomial $P(x) = a_0 + a_1x + a_2x^2 + a_3x^3 + \ldots + a_nx^n$. Clearly we can proceed by differentiating the polynomial expression term by term; from equation (1) it should be clear that the derivative of a sum is just the sum of the derivatives.

Clearly the derivative of the constant term a_0 is just zero. The derivative of a_1x is a_1: $\frac{a_1(x+\delta)-a_1x}{\delta} = \frac{a_1\delta}{\delta} = a_1$. What about a_2x^2? That's a little more work.

$$\begin{aligned}\lim_{\delta\to 0}\frac{a_2(x+\delta)^2 - a_2x^2}{\delta} &= \lim_{\delta\to 0}\frac{a_2(x^2+2x\delta+\delta^2-x^2)}{\delta}\\ &= \lim_{\delta\to 0}\frac{2a_2x\delta + a_2\delta^2}{\delta}\\ &= 2a_2x\end{aligned}$$

because the residual term, $a_2\delta^2/\delta$, vanishes as $\delta \to 0$.

What about a_3x^3, a_4x^4, and all the higher powers of x appearing in $P(x)$? Well, we can apply the Binomial Theorem to write, for any n,

$$a_n((x+\delta)^n - x^n) = a_n(nx^{n-1}\delta + n(n-1)x^{n-2}\delta^2/2! + \ldots)$$

in which all the omitted terms contain δ^3, δ^4, etc. So we can write

$$\begin{aligned}\lim_{\delta\to 0}\frac{a_n((x+\delta)^n - x^n)}{\delta} &= \lim_{\delta\to 0}\frac{a_nnx^{n-1}\delta}{\delta} + \frac{a_nn(n-1)x^{n-2}\delta^2}{2!\delta} + \ldots\\ &= na_nx^{n-1}\end{aligned}$$

because all the terms after the first one vanish as $\delta \to 0$. And so we are led to a simple rule for obtaining the derivative of any polynomial: if $P(x) = \sum_{k=0}^{n} a_kx^k$, then the derivative of $P(x)$ is $P'(x) = \sum_{k=1}^{n} ka_kx^{k-1}$.

Now we have been writing $f'(x)$ to denote the derivative of $f(x)$ because that's how Sir Isaac Newton did it 350 years ago. But there's another way to write the same expression. This alternative notation was invented by Gottfried Wilhelm Leibniz, a German philosopher and mathematician who discovered the rules of the calculus at roughly the same time Newton did. Leibniz expressed the derivative of $f(x)$ this way.

$$y = f(x) \qquad \frac{dy}{dx} = f'(x) \qquad \frac{d}{dx}(x^n) = nx^{n-1}$$

Both systems of notation are still in use today. So you need to be able to work with both of them.

A Geometric Interpretation of Derivatives / Higher Orders

The derivative of a function (x) tells us the rate of change of that function at any point where the function $f(x)$ is differentiable. Geometrically this means that by evaluating $f'(x_0)$ at a point x_0, you can determine the equation of a line that is tangent to the graph of $f(x)$ at the point x_0. Here's an illustration. In Figure 4, $f(x) = x^2 - 4$, $x_0 = 2$, $f(x_0) = y_0 = 0$, $f'(x_0) = 4$, and the equation of the tangent line is $y = 4x - 8$.

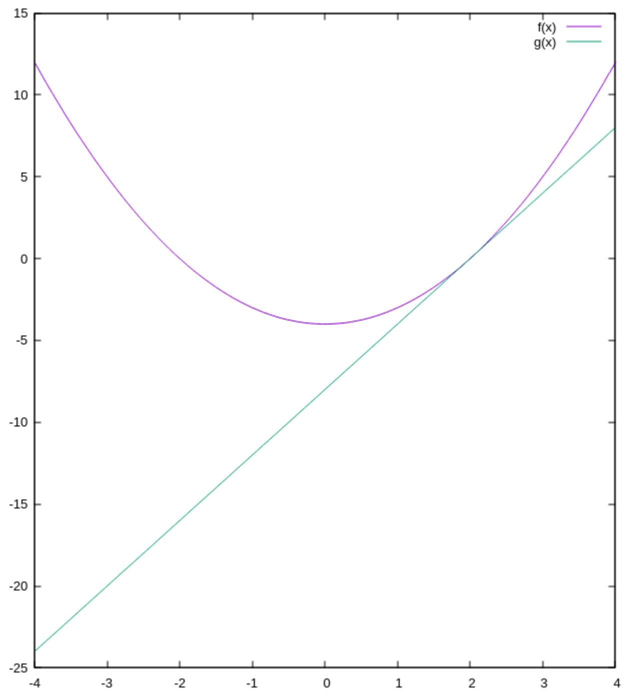

Figure 4: The tangent to $x^2 - 4$ at $(2, 0)$

If we can write down the derivative of a polynomial, can we also take the derivative of a derivative? Indeed we can; the *second derivative* of a polynomial function $P(x)$, also called the *acceleration* of P, is denoted by $P''(x)$, or, using Leibniz' notation, $\frac{d^2}{dx^2}(P(x))$. Applying our differentiation rule twice, it should be obvious that $\frac{d^2}{dx^2}(x^n) = n(n-1)x^{n-2}$.

If we can do it twice, we can do it a third time, and so on. But since the degree of the derivative is one less than the degree of the original polynomial, we'll eventually run out of gas: $\frac{d^n}{dx^n}(x^n) = n!$, a constant. So all the higher derivatives of x^n will be zero, since the derivative of a constant is zero. Is it possible to define functions that have infinitely many non-zero derivatives? Yes! We can construct a *power series* of terms that is, essentially, a polynomial of degree $+\infty$ by defining:

$$f(x) = \sum_{k=0}^{\infty} a_k x^k \tag{2}$$

Such a function is called an *analytic function*. But this is already getting too long, and making sense of (2) would take us too far afield. I'll come back to power series later.