Basic Concepts of Algebra

Algebra is the branch of mathematics that deals with abstract mathematical structures (usually systems of numbers, like the real numbers $\mathbb{R}$), relationships among the elements of such structures ($>, =, <$), and operations that can be performed on those elements ($+, -, *, /$). Algebra is named after a book entitled *Al-Jabr* that was written by Muhammad ibn Musa al-Khwarizmi of Baghdad sometime in the early ninth century (*ca.* 820 A.D.). Al-Khwarizmi is also remembered as the author of a treatise on the symbols used in the Hindu numbering system, which are called Hindu-Arabic numerals today ($0, 1, 2, \ldots, 9$).

Elementary algebra generally deals with relationships and operations on the everyday numbers (natural numbers, and fractions) you learned about in grade school. More advanced algebraic techniques are used to find powers (exponentiation) and roots of numbers, and to solve equations of degree 2, or higher. *Abstract algebra* generalizes the concept of a number to include structures like groups, and rings, and finite fields, and integral domains. This introduction will stick to elementary algebra and everyday numbers like ($1, 2, 3$), ($\frac{1}{2}, \frac{2}{3}, \frac{3}{4}$), and some roots like ($\sqrt{2}$, $\sqrt[3]{5}$), $\sqrt[n]{k}$).

Let's begin our investigation by defining the set of numbers we are dealing with and the symbols we will use to represent those numbers. The numbers we will use will all be elements of the set of real numbers $\mathbb{R}$, which can be visualized as the points on a line that stretches from $-\infty$ to $+\infty$. Real numbers are used to answer questions like "How heavy is it?" or "How long is it?" In everyday experience, we are using real numbers whenever we count things, or when we measure a quantity.

In algebra, when we want to specify a particular number (like five) we will use the Hindu-Arabic numerals to represent it (e.g., 5). When we do not know what numerical value a particular quantity has, we will use a letter, usually x, to represent it. For instance, if we know that multiplying the unknown quantity x by 3 yields the quantity 21 we would write this equation: $3x = 21$.

Now that we understand what sort of numbers we will be dealing with, we should define the relationships we will be working with. There are five of them: $<$, $\leq$, $=$, $\geq$, and $>$, read as "less than", "less than or equal to", "equal to", "greater than or equal to", and "greater than", respectively. All five of these relationships are *transitive*, meaning that if two numbers a and b satisfy the relationship, and if b and c also satisfy the same relationship, then a and c must satisfy the relationship. Symbolically we represent transitivity like this:

$$\begin{array}{ccccc} a < b & \text{and} & b < c & \Rightarrow & a < c \\ a \leq b & \text{and} & b \leq c & \Rightarrow & a \leq c \\ a = b & \text{and} & b = c & \Rightarrow & a = c \\ a \geq b & \text{and} & b \geq c & \Rightarrow & a \geq c \\ a > b & \text{and} & b > c & \Rightarrow & a > c \end{array}$$

where the arrow $\Rightarrow$ signifies implication: "$a < b$ and $b < c$ implies that $a < c$".

Three of the relations, $\leq$, $=$, and $\geq$, are said to be reflexive, meaning that every number a stands in these three relations to itself. Symbolically, $a \leq a$ and $a = a$ and $a \geq a$. The other two relations are not reflexive: the statements $a < a$ and $a > a$ are false for every real number a.

Equality is said to be symmetric: if $a = b$, then $b = a$. The other four relations are not symmetric; they are converse relationships: if $a < b$, then $b > a$, and also if $a \leq b$, then $b \geq a$.

The Four Arithmetic Operations

The four arithmetic operations used in simple algebra are addition (+), subtraction (−), multiplication ($*$, or $\times$), and division (/, or $\div$). Addition is easily visualized by laying a line segment of length a at the end of another such segment of length b and then measuring the total length of the combined segments. Subtraction is the inverse of addition. Multiplication is repeated addition; $n * a =$ the total length of n line segments, each of length a. Division is the operation inverse to multiplication. These operations are characterized by several laws, as follows.

Additive Identity: There is a number 0 such that $a + 0 = 0 + a = a$ for every number a. We call this number 0 the additive identity element in the field of real numbers, $\mathbb{R}$.

Additive Inverse: Every number a has an additive inverse $-a$ such that $a + (-a) = (-a) + a = 0$. Notice that 0 is its own additive inverse, since $0 + 0 = 0$.

Multiplicative Identity: There is a number 1 such that $a * 1 = 1 * a = a$ for every number a. We call this number 1 the multiplicative identity element in $\mathbb{R}$.

Multiplicative Inverse: Every number a, except 0, has a multiplicative inverse $1/a$ such that $a * (1/a) = (1/a) * a = 1$. Notice that 1 is its own multiplicative inverse, since $1 * 1 = 1$.

Associativity: Addition and multiplication are associative. Given any three numbers a, b, and c, $(a + b) + c = a + (b + c) = a + b + c$, and also, $(a * b) * c = a * (b * c) = a * b * c$.

Commutativity: Addition and multiplication are commutative. Given any two numbers a and b, $a + b = b + a$, and also, $a * b = b * a$.

Distributive Property: Multiplication is distributive with respect to addition, which is to say that $a * (b + c) = (a * b + a * c) = a * b + a * c$ for any three numbers a, b, and c.

Exponents and Roots

There is one more operation that is commonly used in elementary algebra, called *exponentiation*. It is defined as repeated multiplication of the same number by itself, and we use a little number, called an exponent, to the right of and slightly above a symbol to indicate how many multiplicands are included in the product. It works like this: $a^1 = a$; $a^2 = a * a$; $a^3 = a^2 * a = a * a * a$; and, in general, $a^n = a^{n-1} * a$. So, for instance, $2^4 = 2 * 2 * 2 * 2 = 16$, and $3^3 = 3 * 3 * 3 = 27$.

From the definition of exponentiation, it should be obvious that $a^m * a^n = a^{m+n}$ for any two natural numbers m, n. We call this *the product rule for exponents*. Similarly, we can figure out that $(a^m)^n = (a^m) * (a^m) * \ldots * (a^m) = a^{m*n}$ where the multiplicand a^m has been repeated n times. This is *the power rule for exponents*.

By definition, if $a \neq 0$, $a^{-1} = 1/a$. Clearly, $a^{-n} = 1/a^n$, and by the product rule for exponents, $a^{m-n} = a^m/a^n$. So we can derive a simple rule about the zero power: $a^0 = a^{m-m} = a^m/a^m = 1$ so long as $a \neq 0$. So $2^0 = 3^0 = \pi^0 = 1$.

We can also use fractions for exponents. By applying the product rule for exponents we see that $a^{1/2} * a^{1/2} = a^{((1/2)+(1/2))} = a^1 = a$. Fractional exponents can also be expressed using the root sign $\sqrt{}$. We see that $(a^{1/2})^2 = a$, which is also the definition of a square root: $(\sqrt{a})^2 = a$. So $a^{1/2} = \sqrt{a}$. Similarly, $a^{1/3} = \sqrt[3]{a}$, and, in general, $a^{1/m} = \sqrt[m]{a}$. Applying the power rule for exponents we can go even farther: $a^{m/n} = \sqrt[m]{a^n}$.

The notion of fractional exponents leads rather naturally to *logarithms*. But that's a topic for another day.

Solving Simple Linear Equations

One of the first things you probably learned in high-school algebra class is how to "solve" a simple linear equation of the form $a*x+b = c*x+d$. The "solution" of such an equation involves taking several steps to reduce it to an equation of the form $x = e$, or, equivalently, $x - e = 0$. Let's quickly review the rules we use to perform the reduction.

Two Equations Can be Added Together: This rule can also be expressed as "The equation is still the same if the same quantity is added to, or subtracted from, both sides." Symbolically it looks like this.

$$\begin{aligned} a*x+b &= c*x+d \\ -b &= -b \\ a*x+0 &= c*x+(d-b) \\ a*x &= c*x+(d-b) \end{aligned}$$

You can think of this process as adding the first equation (which was given) to the second equation (which must be true, because equality is reflexive) to produce the third equation, and then to remove the additive identity element on the left hand side to arrive at the fourth equation (because adding 0 to any number leaves it unchanged).

But now that we have all the constant terms on the right hand side of the equation, we can use the same trick we used above to move every occurrence of the unknown quantity x to the left hand side.

$$\begin{aligned} a*x &= c*x+(d-b) \\ -c*x &= -c*x \\ a*x-c*x &= 0*x+(d-b) \\ (a-c)*x &= (d-b) \end{aligned}$$

We start with the transformed equation. We subtract cx from both sides. Recognizing that $c*x - c*x = (c-c)*x$ and also that $a*x - c*x = (a-c)*x$ (distributive property), and that $(c-c)*x = 0*x = 0$ (additive inverse), we end with x appearing only on the left hand side. Now we need to apply another rule.

Two Equations Can be Multiplied Together: This rule can also be expressed as "The equation is still the same if both sides are multiplied by, or divided by, the same non-zero number." Symbolically it looks like this: If $a \neq 0$, $ax = b$ means the same thing as $x = a/b$, or, equivalently, $ax = b$ implies that $x = a/b$.

So let's continue the example from above. Assume that $a - c \neq 0$.

$$
\begin{aligned}
(a-c) * x &= (d-b) \\
\frac{1}{(a-c)} &= \frac{1}{(a-c)} \\
\frac{1}{(a-c)} * (a-c) * x &= \frac{1}{(a-c)} * (d-b) \\
1 * x &= \frac{(d-b)}{(a-c)} \\
x &= \frac{d-b}{a-c}
\end{aligned}
$$

First we write the transformed equation from above. Then we use the reflexive property of equality to assert that $1/(a-c) = 1/(a-c)$. Next we use the new rule to multiply the first equation by the second one. We know that any number times its multiplicative inverse is just 1, or unity. And since multiplying any number times one gives us the same number back, we arrive at the final result, $x = (d-b)/(c-a)$. We have "solved" the general linear equation!

There's a lot more to algebra than linear equations, or equations of the first degree. For instance, we may have two equations in two unknowns: $ax + by = c$ and $dx + ey = f$. Or the equation may be a quadratic equation, also known as an equation of the second degree: $ax^2 + bx + c = 0$. Or the equation may be a cubic (third degree) or quartic (fourth degree) equation. But all those things are best left for another day. Some of them can get pretty hairy. Good-bye for now.