# Introduction to Transfinite Numbers

Georg Cantor, a German mathematician who lived in the late nineteenth century, famously asked a basic question about arithmetic: what are we really doing when we count things? The answer he derived gave rise to a branch of modern mathematics known as *transfinite arithmetic*. Let's follow Cantor's logic a little ways, and see where it takes us.

So let's imagine that we don't know how to count, but we have some apples and some oranges and we wonder, are there more apples? Or are there more oranges? What would we do? The obvious approach would be to begin forming pairs of apples and oranges. After forming all the pairs we could, if we still have some apples left, we would conclude that we have more apples than oranges. If oranges are left unpaired, we have more oranges than apples. And if there are neither apples nor oranges left over, we have an equal number of each.

Cantor formalized this insight with the notion of a one-to-one correspondence. We say that two sets $A = \{a_1, a_2, a_3, \ldots, a_n\}$ and $B = \{b_1, b_2, b_3, \ldots, b_m\}$ are equinumerous, or have the same number of elements, if and only if we can establish a one-to-one correspondence between the two sets: when we form the pairs $(a_1, b_1), (a_2, b_2), (a_3, b_3), \ldots, (a_n, b_m)$ we come out even, with no excessive elements from either $A$ or $B$ left over. If there are elements $a$ left over,$A$ has more elements than $B$. And vice-versa. So the notion of a one-to-one correspondence leads us naturally to the relationships $<, =$, and $>$, obtaining among the natural numbers.

Before we tackle Kantor's theory of transfinite numbers, we should review a few simple concepts from set theory, just to be sure we have them firmly in mind. A set $A$ is merely a collection of objects. We don't need to know what kind of objects they are, they're just random things. A particular thing is said to be in the set $A$, or to be an element of $A$, if it is included in $A$. Symbolically the statement "$a$ is an element of the set $A$" is represented by $a \in A$. Conversely, if $a$ is not an element of $A$ we write $a \notin A$.

A couple of examples should make this clear. $7 \in \{1, 2, 5, 7, 11, 13\}$, but $8 \notin \{1, 2, 5, 7, 11, 13\}$. Similarly, $c \in \{a, b, c, d, e, f\}$, but $k \notin \{a, b, c, d, e, f\}$.

We can add two sets together by combining all the elements of both sets into one new set. We call the new set $C$ the *union* of the original sets $A$ and $B$. Symbolically we write $C = A \cup B$. We can also separate the elements common to both $A$ and $B$ into a new set $C$, called the *intersection* of $A$ and $B$. Symbolocally we say "$C$ is the intersection of $A$ and $B$" by writing $C = A \cap B$. Thus, for instance, $\{1, 2, 3\} \cup \{5, 7\} = \{1, 2, 3, 5, 7\}$, and $\{a, b, c, d, e\} \cap \{d, e, f, g, h\} = \{d, e\}$.

We define the null set (or empty set) $\emptyset$ as the set with no elements. By convention, $\emptyset \in A$ for every set A. Also by convention, $A \cup \emptyset = A$ for every set $A$, and $A \cap \emptyset = \emptyset$ for any $A$.

A set that contains some, but not all, of the elements of $A$ is said to be a *proper subset* of $A$. $A$ is said to be the superset of any one of its proper subsets. $A$ is an improper subset of itself, and also an improper superset of itself. Symbolically we write $B \subset A$ to denote "B is a proper subset of A" or "B is properly contained in A"; $A \supset B$ means the same thing, but we say "A is a superset of B" or "A properly contains B". For any set $A$ we can write $A \subseteq A$ and $A \supseteq A$ (improper subset and superset, respectively).

With these notational conventions established, we can proceed with Cantor to infinity, and beyond. Cantor asked himself “What distinguishes a set with infinitely many elements from a finite set?” The answer, he said, is that a set is infinite if and only if its elements can be placed in one-to-one correspondence with a proper subset of itself. Thus the set of all natural numbers $\mathbb{N}$ is infinite because it can be placed in one-to-one correspondence with the even numbers ($n \leftrightarrow 2n$), or with multiples of $5$ ($1 \leftrightarrow 5$, $2 \leftrightarrow 10$, $3 \leftrightarrow 15$, $\ldots$), etc.

Is it possible to count all the items in some infinite set, such as the rational fractions? Yes, Cantor said. If the elements of two infinite sets can be placed in a one-to-one correspondence, then they have the same number of elements, and are said to be of equal *cardinality*. Now intuitively there are a lot more fractions than there are whole numbers. But Cantor argued otherwise, using a clever diagram.

$$
\begin{array}{cccccccc}
\frac{1}{1} & & \frac{1}{2} & \rightarrow & \frac{1}{3} & & \frac{1}{4} & \cdots \\
\downarrow & \nearrow & & \swarrow & & \nearrow & & \swarrow \\
\frac{2}{1} & & \frac{2}{2} & & \frac{2}{3} & & \frac{2}{4} & \cdots \\
 & \swarrow & & \nearrow & & \swarrow & & \\
\frac{3}{1} & & \frac{3}{2} & & \frac{3}{3} & & \frac{3}{4} & \cdots \\
\downarrow & \nearrow & & \swarrow & & & & \\
\frac{4}{1} & & \frac{4}{2} & & \frac{4}{3} & & \frac{4}{4} & \cdots \\
\vdots & \swarrow & \vdots & & \vdots & & \vdots &
\end{array}
$$

Clearly the table will contain all the rational fractions if it’s extended all the way. And just as clearly the items in the table can be placed in one-to-one correspondence with the natural numbers by just zig-zagging back and forth, as shown by the arrows. So there are no more fractions than there are natural numbers; $\mathbb{N}$ and $\mathbb{Q}$ have the same cardinality.

Cantor was Jewish, so he chose the Hebrew letter aleph ($\aleph$) to denote an infinitely large cardinal number. He called the first one, the cardinality of $\mathbb{N}$, $\aleph_0$. “Are there any more?”, he asked. Soon a reply came back: yes, there are.

Cantor proceeded by considering the *power set* of $\aleph_0$. The power set $P$ of a finite set is just the set of all the sets that can be formed from the original set $A$, remembering that $\emptyset$ and $A$ are to be counted as elements of $A$. For instance, we see that if $A = \{a, b, c\}$, the power set of $A$ is given by $\{\{\emptyset\}, \{a\}, \{b\}, \{c\}, \{a, b\}, \{a, c\}, \{b, c\}, \{a, b, c\}\}$.

Counting the elements of $A$ and $P$, we see that there are $3$ elements (not null) in $A$, and there are $8 = 2^3$ elements in $P$. And it can easily be proved, by mathematical induction, that this relationship holds for any finite set $A$: if the cardinality of $A$ is $n$, the cardinality of $A$’s power set $P_A$ is $2^n$. Cantor extrapolated this result to the infinite set $\mathbb{N}$ and asserted that the power set of $\mathbb{N}$ is of cardinality $2^{\aleph_0} = \aleph_1$. Cantor used the same sort of zig-zag proof that we used above (to prove that $\mathbb{Q}$ and $\mathbb{N}$ are equinumerous) to show that the natural numbers cannot possibly be placed in a one-to-one correspondence with the elements of $P_{\mathbb{N}}$, and he concluded that $\aleph_1 > \aleph_0$. He further asserted the existence of infinitely many cardinal numbers $\aleph_2, \aleph_3, \aleph_4, \ldots$, each one defined by reference to successive power sets of increasing infinities, and all satisfying the equation $\aleph_n = 2^{\aleph_{n-1}}$.

Cantor's theories about transfinite numbers were very controversial when he first introduced them. Some of his contemporaries praised him as a visionary. Others condemned him as a lunatic. Later in life he suffered from recurrent bouts of depression, which eventually led to his confinement in a sanitorium, where he died. Today his work is widely recognized as a seminal contribution to set theory, which is considered foundational for all of mathematics.

## Transfinite Ordinals and the Continuum Hypothesis

Cantor also constructed a class of transfinite ordinal numbers, but I'm not going to dive into those today. Instead, I'd like to talk about denumerable sets, and non-denumerable sets, and the so-called *continuum hypothesis*.

We say that any set $A$ whose elements can be placed in one-to-one correspondence with the natural numbers is *denumerable*, or countable, or countably infinite. If the elements of $A$ cannot be placed in on-to-one correspondence with the natural numbers, $A$ is said to be non-denumerable, or uncountable, or uncountably infinite. We already know that the rational numbers $\mathbb{Q}$ are countable. Cantor also showed that the set of algebraic numbers (i.e., all the roots of polynomial equations with rational coefficients) is denumerable. Are there any concrete examples of sets that are non-denumerable? Yes, there are. The set of ordinary real numbers $\mathbb{R}$ is one such example. We can prove this with a very simple argument. The proof is by contradiction.

Let us suppose that we have placed all the real numbers $r \ni 0 < r < 1$ (read "r such that 0 is less than r which is less than one") in one-to-one correspondence with the natural numbers. Then, since every such real number can be expressed as a decimal fraction, we can create a list like this:

$$\begin{array}{ccl} 1 & \leftrightarrow & 0.d_1\, d_2\, d_3\, d_4\, d_5\, d_6\, d_7\, \ldots \\ 2 & \leftrightarrow & 0.d_1\, d_2\, d_3\, d_4\, d_5\, d_6\, d_7\, \ldots \\ 3 & \leftrightarrow & 0.d_1\, d_2\, d_3\, d_4\, d_5\, d_6\, d_7\, \ldots \\ 4 & \leftrightarrow & 0.d_1\, d_2\, d_3\, d_4\, d_5\, d_6\, d_7\, \ldots \\ 5 & \leftrightarrow & 0.d_1\, d_2\, d_3\, d_4\, d_5\, d_6\, d_7\, \ldots \\ 6 & \leftrightarrow & 0.d_1\, d_2\, d_3\, d_4\, d_5\, d_6\, d_7\, \ldots \\ 7 & \leftrightarrow & 0.d_1\, d_2\, d_3\, d_4\, d_5\, d_6\, d_7\, \ldots \\ \vdots & \vdots & \qquad\quad \vdots \end{array}$$

where the $d_i$ represent the decimal digits of each real number. But such a list cannot possibly exist because, given the list we can construct a new number $r = 0.r_1\, r_2\, r_3\, r_4\, r_5\, r_6\, r_7\, \ldots$ that differs from the first number in the first decimal place, that differs from the second number in the second decimal place, and so forth. The new number is a real number, but it can't be on the list because it's different from the $n$th real number in the $n$th decimal place. Since the list does not contain all the real numbers $r \ni 0 < r < 1$, we conclude that the set of real numbers $\mathbb{R}$ is not countable.

Cantor was also able to prove with a very clever argument that the field of complex numbers can be placed in one-to-one correspondence with the real number line (i.e., $\mathbb{C}$ and $\mathbb{R}$ are equinumerous). He also conjectured, but could not prove, that there is no transfinite cardinal number $\beth \ni \aleph_0 < \beth < 2^{\aleph_0}$. This conjecture became known as the Continuum Hypothesis, and for 85 years it was one of the most famous unsolved problems in all of mathematics. Finally, in 1963, a mathematician named Paul Cohen was able to prove that the Continuum Hypothesis can neither be proved nor disproved by reasoning from the standard axioms of set theory. In other words, the Continuum Hypothesis is independent of the axioms of set theory, and one may assume it is either true or false without creating a contradiction.